

# CHARACTERIZATIONS OF $I$ -SEMIREGULAR AND $I$ -SEMIPERFECT RINGS

Yongduo Wang

Department of Applied Mathematics, Lanzhou University of Technology  
Lanzhou 730050, P. R. China  
E-mail: ydwang@lut.cn

**Abstract** Let  $M$  be a left module over a ring  $R$  and  $I$  an ideal of  $R$ . We call  $(P, f)$  a (locally)projective  $I$ -cover of  $M$  if  $f$  is an epimorphism from  $P$  to  $M$ ,  $P$  is (locally)projective,  $\text{Ker} f \subseteq IP$ , and whenever  $P = \text{Ker} f + X$ , then there is a projective summand  $Y$  of  $P$  in  $\text{Ker} f$  such that  $P = Y \oplus X$ . This definition generalizes (locally)projective covers. We characterize  $I$ -semiregular and  $I$ -semiperfect rings which are defined by Yousif and Zhou [19] using (locally)projective  $I$ -covers in section 2 and 3.  $I$ -semiregular and  $I$ -semiperfect rings are characterized by projectivity classes in section 4. Finally, the notion of  $I$ -supplemented modules are introduced and  $I$ -semiregular and  $I$ -semiperfect rings are characterized by  $I$ -supplemented modules. Some well known results are obtained as corollaries.

**Keywords:** Semiregular, Semiperfect, Locally projective modules, Projectivity class

## 1. Introduction and Preliminaries

It is well known that (locally) projective covers, projectivity classes and supplemented modules play important roles in characterizing semiperfect and semiregular rings. Recently, some authors had worked with various extensions of these rings (see for examples [2, 9, 11, 19, 20]). As generalizations of semiregular rings and semiperfect rings, the notions of  $I$ -semiregular rings and  $I$ -semiperfect rings were introduced by Yousif and Zhou [19]. Our purposes of this paper as follows: (1) characterize  $I$ -semiregular and  $I$ -semiperfect rings by defining (locally) projective  $I$ -covers in section 2 and 3; (2) characterize  $I$ -semiregular rings and  $I$ -semiperfect

rings in term of projectivity classes of modules in section 4. (3) chracterize  $I$ -semiregular rings and  $I$ -semiperfect rings by defining  $I$ -supplemented modules in section 5.

Let  $R$  be a ring and  $I$  an ideal of  $R$ ,  $M$  a module and  $S \leq M$ .  $S$  is called *small* in  $M$  (notation  $S \ll M$ ) if  $M \neq S + T$  for any proper submodule  $T$  of  $M$ . As a proper generalization of small submodules, the concept of  $\delta$ -small submodules was introduced by Zhou in [20]. Let  $N \leq M$ .  $N$  is said to be  $\delta$ -small in  $M$  if, whenever  $N + X = M$  with  $M/X$  singular, we have  $X = M$ .  $\delta(M) = \text{Rej}_M(\wp) = \cap \{N \leq M \mid M/N \in \wp\}$ , where  $\wp$  be the class of all singular simple modules. We also recall that a pair  $(P, f)$  is called a (locally) projective  $(\delta-)$ cover of  $M$  if  $P$  is (locally) projective,  $f$  is an epimorphism from  $P$  to  $M$  such that  $\text{Ker} f \ll P(\text{Ker} f \ll_\delta P)$ . An element  $m$  of  $M$  is called  *$I$ -semiregular* [2] if there exists a decomposition  $M = P \oplus Q$  where  $P$  is projective,  $P \subseteq Rm$  and  $Rm \cap Q \subseteq IM$ .  $M$  is called an  *$I$ -semiregular module* if every element of  $M$  is  $I$ -semiregular.  $R$  is called  *$I$ -semiregular* if  ${}_R R$  is an  $I$ -semiregular module. Note that  $I$ -semiregular rings are left-right symmetric and  $R$  is  $(\delta-)$  semiregular if and only  $R$  is  $(\delta({}_R R)-) J(R)$ -semiregular.  $M$  is called an  *$I$ -semiperfect module* [11] if for every submodule  $K$  of  $M$ , there is a decomposition  $M = A \oplus B$  such that  $A$  is projective,  $A \subseteq K$  and  $K \cap B \subseteq IM$ .  $R$  is called  *$I$ -semiperfect* if  ${}_R R$  is an  $I$ -semiperfect module. Note that  $R$  is  $(\delta-)$  semiperfect if and only  $R$  is  $(\delta({}_R R)-) J(R)$ -semiperfect. For other standard definitions we refer to [3, 10, 13].

In this note all rings are associative with identity and all modules are unital left modules unless specified otherwise. Let  $R$  be a ring and  $M$  a module. We use  $\text{Rad}(M)$ ,  $\text{Soc}(M)$ ,  $Z(M)$  to indicate the Jacobson radical, the socle, the singular submodule of  $M$  respectively.  $J(R)$  is the radical of  $R$  and  $I$  is an ideal of  $R$ .

## 2. $I$ -semiregular( $I$ -semiperfect) rings and projective $I$ -covers

In this section, we introduce the notion of PSD submodules of modules and use this to define projective  $I$ -covers which are a generalization of some well-known

projective covers. Characterizations of  $I$ -semiregular and  $I$ -semiperfect rings are given by projective  $I$ -covers. We begin this section with the following definitions.

**Definition 2.1.** Let  $I$  be an ideal of  $R$  and  $N \leq M$ .  $N$  is PSD in  $M$  if there exists a projective summand  $S$  of  $M$  such that  $S \leq N$  and  $M = S \oplus X$  whenever  $N + X = M$  for any submodule  $X \leq M$ .  $M$  is PSD for  $I$  if any submodule of  $IM$  is PSD in  $M$ .  $R$  is a left PSD ring for  $I$  if any finitely generated free left  $R$ -module is PSD for  $I$ .

**Lemma 2.2.** *Let  $N$  be a direct summand of  $M$  and  $A \leq N$ . Then  $A$  is PSD in  $M$  if and only if  $A$  is PSD in  $N$ .*

*Proof.* “ $\Rightarrow$ ” Since  $N$  is a direct summand of  $M$ ,  $M = N \oplus K$  for some submodule  $K \leq M$ . Suppose that  $N = A + X$ ,  $X \leq N$ , then  $M = A + (X \oplus K)$ . Since  $A$  is PSD in  $M$ , there is a projective direct summand  $Y$  of  $M$  such that  $Y \leq A$  and  $M = Y \oplus X \oplus K$ , and hence  $N = N \cap M = X \oplus Y$ .

“ $\Leftarrow$ ” Let  $M = A + L$ ,  $L \leq M$ . Then  $N = N \cap M = A + N \cap L$ . Since  $A$  is PSD in  $N$ , there is a projective summand  $K$  of  $N$  with  $K \leq A$  such that  $N = K \oplus (N \cap L)$ . It is easy to see that  $K \cap L = 0$ . Next we only show that  $M = K + L$ . Let  $m \in M$ , then  $m = a + l$ ,  $a \in A$ ,  $l \in L$ . Since  $a = k + s$ ,  $k \in K$ ,  $s \in N \cap L$ ,  $m = k + s + l$ . Note that  $s + l \in L$ , so  $m \in K + L$ , and hence  $M = K + L$ , as required. □

**Corollary 2.3.** *Let  $M$  be a  $R$ -module. If  $M$  is PSD for an ideal  $I$  of  $R$ , then any direct summand of  $M$  is PSD for  $I$ .*

**Corollary 2.4.** *A ring  $R$  is a left PSD ring for an ideal  $I$  if and only if any finitely generated projective left  $R$ -module is PSD for  $I$ .*

**Proposition 2.5.** *Let  $M = M_1 \oplus M_2$ . If  $N_1$  is PSD in  $M_1$  and  $N_2$  is PSD in  $M_2$ , then  $N_1 \oplus N_2$  is PSD in  $M$ .*

*Proof.* Let  $M = N_1 \oplus N_2 + L$ ,  $L \leq M$ . Since  $N_1$  is PSD in  $M_1$ ,  $N_1$  is PSD in  $M$  by Lemma 2.2. Thus there is a projective summand  $S_1$  of  $M$  with  $S_1 \subseteq N_1$  such

that  $M = S_1 \oplus (N_2 + L)$ . Similarly, there exists a projective summand  $S_2$  of  $M$  with  $S_2 \subseteq N_2$  such that  $M = S_1 \oplus S_2 + L$ . The rest is obvious.  $\square$

**Definition 2.6.** A pair  $(P, f)$  is called a projective  $I$ -cover of  $M$  if  $P$  is projective,  $f$  is an epimorphism from  $P$  to  $M$  such that  $\text{Ker} f \leq IP$ , and  $\text{Ker} f$  is PSD in  $P$ .

It is easy to see that a module  $M$  has a projective  $\delta({}_R R)$ -cover (projective  $J(R)$ -cover, respectively) if and only if  $M$  has a projective  $\delta$ -cover (projective cover, respectively) by [1, Proposition 3.6].

**Proposition 2.7.** *If each  $f_i : P_i \rightarrow M_i, (i = 1, 2, \dots, n)$  is a projective  $I$ -cover, then  $\oplus_{i=1}^n f_i : \oplus_{i=1}^n P_i \rightarrow \oplus_{i=1}^n M_i$  is a projective  $I$ -cover.*

*Proof.* By Proposition 2.5 and the definition of projective  $I$ -covers.  $\square$

**Lemma 2.8.** *Let  $I$  be an ideal of  $R$  and  $f : P \rightarrow M$  a projective  $I$ -cover. If  $Q$  is projective and  $g : Q \rightarrow M$  is epic. Then there are decompositions  $P = A \oplus B$  and  $Q = X \oplus Y$  such that*

- (1)  $A \cong X$ ;
- (2)  $f|_A : A \rightarrow M$  is a projective  $I$ -cover;
- (3)  $g|_X : X \rightarrow M$  is a projective  $I$ -cover;
- (4)  $B \subseteq \text{Ker} f, Y \subseteq \text{Ker} g$ .

*Proof.* Since  $Q$  is projective, there is a homomorphism  $h : Q \rightarrow P$  such that  $fh = g$ , and so  $P = h(Q) + \text{Ker} f$ . Since  $\text{Ker} f$  is PSD in  $P$ , there is a direct summand  $B$  of  $P$  such that  $P = A \oplus B$  with  $A = h(Q), B \subseteq \text{Ker} f$ . We shall show that  $f|_A : A \rightarrow M$  is a projective  $I$ -cover. It is clear that  $\text{Ker} f|_A = A \cap \text{Ker} f \subseteq A \cap IP = IA$ . Let  $\text{Ker} f|_A + L = A, L \leq A$ . Then  $P = \text{Ker} f|_A + L \oplus B = \text{Ker} f + L \oplus B$ . Since  $\text{Ker} f$  is PSD in  $P$ , there is a direct summand  $K$  of  $P$  ( $K \subseteq \text{Ker} f$ ) such that  $P = L \oplus K \oplus B$ , and hence  $A = A \cap P = A \cap (L \oplus K \oplus B) = L \oplus (A \cap (K \oplus B))$ . It is easy to see  $A \cap (K \oplus B) \subseteq \text{Ker} f|_A$ , so  $\text{Ker} f|_A$  is PSD in  $A$ . Thus  $f|_A : A \rightarrow M$  is a projective  $I$ -cover. Since  $A$  is projective,  $h : Q \rightarrow A$  splits, and hence there is a homomorphism  $q : A \rightarrow Q$  such that  $hq = 1_A$ . So  $Q = X \oplus Y, X = q(A), Y = \text{Ker} h$ .

Since  $X = q(A)$ ,  $A \cong X$ . Next we show that  $g|_X: X \rightarrow M$  is a projective  $I$ -cover. Since  $g(X) = fh(X) = fh(X + Y) = fh(Q) = M$ ,  $g|_X: X \rightarrow M$  is epic. We have  $\text{Ker } g|_X = q(\text{Ker } f|_A) \subseteq q(IA) = IX$ . Now we only show that  $\text{Ker } g|_X$  is PSD in  $X$ . Assume that  $X = \text{Ker } g|_X + N$ ,  $X \leq N$ , then  $A = h(\text{Ker } g|_X) + h(N) = \text{Ker } f|_A + h(N)$ . Since  $\text{Ker } f|_A$  is PSD in  $A$ , there is a direct summand  $Z$  of  $A$  ( $Z \subseteq \text{Ker } f|_A$ ) such that  $A = Z \oplus h(N)$ . We know that  $h|_X: X \rightarrow A$  is isomorphic, so  $X = h|_X^{-1}(Z) \oplus N$ . It is easy to verify that  $h|_X^{-1}(Z) \subseteq \text{Ker } g|_X$ , as required.  $\square$

**Lemma 2.9.** *Let  $I$  be an ideal of a ring  $R$ ,  $M$  is a projective  $R$ -module and  $N \leq M$ . Consider the following conditions:*

- (1)  $M/N$  has a projective  $I$ -cover.
- (2)  $M = Y \oplus X$ ,  $Y \leq N$ ,  $X \cap N \leq IM$ .

*Then (1)  $\Rightarrow$  (2), (2)  $\Rightarrow$  (1) if  $M$  is PSD for  $I$ .*

*Proof.* “(1)  $\Rightarrow$  (2)” Let  $f: P \rightarrow M/N$  be a projective  $I$ -cover and  $g: M \rightarrow M/N$  be the canonical epimorphism. By Lemma 2.8,  $M = Y \oplus X$ ,  $Y \subseteq \text{Ker } g = N$  and  $\text{Ker } g|_X: X \rightarrow M/N$  is a projective  $I$ -cover, and so  $\text{Ker } g|_X = X \cap N \subseteq IX \subseteq IM$ .

“(2)  $\Rightarrow$  (1)” Let  $f: X \rightarrow M/N$  with  $f(x) = x + N$ . It is easy to see that  $f: X \rightarrow M/N$  is a projective  $I$ -cover by Lemma 2.2 and assumptions.  $\square$

With Lemma 2.9, we can give the following characterization of  $I$ -semiregular rings related to projective  $I$ -covers.

**Theorem 2.10.** *Let  $I$  be an ideal of  $R$ . Consider the following conditions:*

- (1) Every finitely presented  $R$ -module has a projective  $I$ -cover.
- (2) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a projective  $I$ -cover.
- (3) Every cyclically presented left  $R$ -module has a projective  $I$ -cover.
- (4)  $R$  is  $I$ -semiregular.

*Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4), (4)  $\Rightarrow$  (1) if  $R$  is a left PSD ring for  $I$ .*

*Proof.* By Lemma 2.9, the rest is similar to [1, Theorem 3.11].  $\square$

If  $I = \delta({}_R R)$  or  $J(R)$ , then  $R$  is a left PSD ring for  $I$ , and hence Theorem 2.10 gives the characterizations of  $\delta$ -semiregular rings [20] and semiregular rings [10]. Since if  $R$  is  $Z({}_R R)$ -semiregular, then  $Z({}_R R) = J(R) \subseteq \delta({}_R R)$ , we have the following result.

**Corollary 2.11.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $Z({}_R R)$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a projective  $Z({}_R R)$ -cover.
- (3) Every finitely presented  $R$ -module has a projective  $Z({}_R R)$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a projective  $Z({}_R R)$ -cover.

Since if  $I \leq \text{Soc}({}_R R)$ , then  $R$  is a left PSD ring for  $I$ , and hence we have

**Corollary 2.12.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $\text{Soc}({}_R R)$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a projective  $\text{Soc}({}_R R)$ -cover.
- (3) Every finitely presented  $R$ -module has a projective  $\text{Soc}({}_R R)$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a projective  $\text{Soc}({}_R R)$ -cover.

Next we shall consider the characterizations of  $I$ -semiperfect rings.

**Theorem 2.13.** *Let  $I$  be an ideal of  $R$ . Consider the following conditions:*

- (1) Every finitely generated  $R$ -module has a projective  $I$ -cover.
- (2) Every factor module of  ${}_R R$  has a projective  $I$ -cover.
- (3) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a projective  $I$ -cover.
- (4)  $R$  is  $I$ -semiperfect.
- (5) Every simple  $R$ -module has a projective  $I$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a projective  $I$ -cover.

Then  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$  and  $(1) \Rightarrow (5) \Rightarrow (6)$ ,  $(4) \Rightarrow (1)$  if  $R$  is a left PSD ring for  $I$ ; and  $(6) \Rightarrow (4)$  if  $I$  is PSD in  ${}_R R$ .

*Proof.* Similar to [1, Theorem 4.8] by Lemma 2.9.  $\square$

When  $I = \delta({}_R R)$  or  $\text{Soc}({}_R R)$  or  $J(R)$ , Theorem 2.13 gives the characterizations of  $(\delta\text{-}, \text{Soc}({}_R R)\text{-})$  semiperfect rings (See [20], [11], [10]).

**Corollary 2.14.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $Z({}_R R)$ -semiperfect.
- (2) Every finitely generated  $R$ -module has a projective  $Z({}_R R)$ -cover.
- (3) Every factor module of  ${}_R R$  has a projective  $Z({}_R R)$ -cover.
- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a projective  $Z({}_R R)$ -cover.
- (5) Every simple  $R$ -module has a projective  $Z({}_R R)$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a projective  $Z({}_R R)$ -cover.

### 3. $I$ -semiregular( $I$ -semiperfect) rings and locally projective $I$ -covers

Ding and Chen [5] and Xue [18] gave some characterizations of rings by locally projective covers. Inspired by those, we introduce the notion of locally projective  $I$ -covers and use it to characterize  $I$ -semiregular and  $I$ -semiperfect rings in this section. Firstly, we recall some definitions and facts. A module  $P$  is called locally projective [18, 21] in case it satisfies any of the following equivalent condition: (a) if  $A$  and  $B$  are modules,  $g : A \rightarrow B$  is an epimorphism and  $f : P \rightarrow B$  is a homomorphism then for every finitely generated (cyclic) submodule  $P_0$  of  $P$  there is a homomorphism  $h : P \rightarrow A$  such that  $f|_{P_0} = gh|_{P_0}$ ; (b) if  $M$  is a module and  $f : M \rightarrow P$  is an epimorphism then for every finitely generated (cyclic) submodule  $P_0$  of  $P$  there is a homomorphism  $h : P \rightarrow M$  such that  $fh|_{P_0} = 1|_{P_0}$ . Clearly, every finitely generated locally projective module is projective. The following facts are well known. (1) a direct sum of locally projective modules is locally projective; (2) any direct summand of a locally projective module is locally projective; (3) if

$P$  is a locally projective module, then (1)  $\text{Rad}(P) = J(R)P$ ; (2) if  $\text{Rad}(P) = P$ , then  $P = 0$ . We also recall that a pair  $(P, f)$  is called a locally projective  $(\delta)$ -cover of  $M$  if  $P$  is locally projective,  $f$  is an epimorphism from  $P$  to  $M$  such that  $\text{Ker}f \ll P$  ( $\text{Ker}f \ll_\delta P$ ).

**Definition 3.1.** A pair  $(P, f)$  is called a locally projective  $I$ -cover of  $M$  if  $P$  is locally projective,  $f$  is an epimorphism from  $P$  to  $M$  such that  $\text{Ker}f \leq IP$ , and  $\text{Ker}f$  is PSD in  $P$ .

It is easy to see that a module  $M$  has a locally projective 0-cover if and only if  $M$  is locally projective.

**Proposition 3.2.** *If each  $f_i : P_i \rightarrow M_i, (i = 1, 2, \dots, n)$  is a locally projective  $I$ -cover, then  $\oplus_{i=1}^n f_i : \oplus_{i=1}^n P_i \rightarrow \oplus_{i=1}^n M_i$  is a locally projective  $I$ -cover.*

**Proposition 3.3.** *A module  $M$  has a locally projective  $J(R)$ -cover if and only if  $M$  has a locally projective cover.*

*Proof.* “ $\Leftarrow$ ” is clear.

“ $\Rightarrow$ ” Let  $f : P \rightarrow M$  be a locally projective  $J(R)$ -cover. Then  $\text{Ker}f \subseteq J(R)P$ ,  $\text{Ker}f$  is PSD in  $P$ . Next we shall show that  $\text{Ker}f \ll P$ . Let  $\text{Ker}f + L = P, L \leq M$ . Since  $\text{Ker}f$  is PSD in  $P$ , there is a summand  $K$  of  $P$  with  $K \leq \text{Ker}f$  such that  $K \oplus L = P$ . Since  $\text{Rad}(K) \oplus \text{Rad}(L) = \text{Rad}(P)$  and  $K \subseteq \text{Rad}(P)$ ,  $\text{Rad}(K) = K$ . Since  $K$  is locally projective,  $K = 0$ , and hence  $L = P$ , as desired.  $\square$

The following lemma is a key result of this section.

**Lemma 3.4.** *Let  $f : P \rightarrow M$  be a locally projective  $I$ -cover. If  $M$  is finitely generated, then  $f : P \rightarrow M$  is a projective  $I$ -cover.*

*Proof.* It suffices to prove that  $P$  is projective. Since  $f : P \rightarrow M$  is a locally projective  $I$ -cover and  $M$  is finitely generated, there is a finitely generated submodule  $P_0$  of  $P$  such that  $P_0 + \text{Ker}f = P$ . Note that  $\text{Ker}f$  is PSD in  $P$ , there is a projective summand  $K$  of  $P$  with  $K \subseteq \text{Ker}f$  such that  $P_0 \oplus K = P$ . Since  $P$  is locally



projective,  $P_0$  is locally projective. Since  $P_0$  is finitely generated,  $P_0$  is projective. Thus  $P$  is projective, as required.  $\square$

**Theorem 3.5.** *Let  $I$  be an ideal of  $R$ . Consider the following conditions:*

- (1) *Every finitely presented  $R$ -module has a locally projective  $I$ -cover.*
- (2) *For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $I$ -cover.*
- (3) *Every cyclically presented left  $R$ -module has a locally projective  $I$ -cover.*
- (4)  *$R$  is  $I$ -semiregular.*

*Then  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ ,  $(4) \Rightarrow (1)$  if  $R$  is a left PSD ring for  $I$ .*

*Proof.* “ $(1) \Rightarrow (2) \Rightarrow (3)$ ” are clear.

“ $(3) \Rightarrow (4)$ ” It follows by Lemma 3.4 and Theorem 2.10.

“ $(4) \Rightarrow (1)$ ” is clear by Theorem 2.10.  $\square$

If  $I = \delta({}_R R)$  or  $J(R)$ , then  $R$  is a left PSD ring for  $I$ .

**Corollary 3.6.** *The following conditions are equivalent for a ring  $R$ .*

- (1)  *$R$  is semiregular.*
- (2) *Every finitely presented  $R$ -module has a locally projective cover.*
- (3) *For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective cover.*
- (4) *Every cyclically presented left  $R$ -module has a locally projective cover.*

**Corollary 3.7.** *The following conditions are equivalent for a ring  $R$ .*

- (1)  *$R$  is  $\delta$ -semiregular.*
- (2) *Every finitely presented  $R$ -module has a locally projective  $\delta$ -cover.*
- (3) *For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $\delta$ -cover.*
- (4) *Every cyclically presented left  $R$ -module has a locally projective  $\delta$ -cover.*

*Proof.* It follows by Theorem 2.10 and Lemma 3.4.  $\square$

Since if  $R$  is  $Z({}_R R)$ -semiregular, then  $Z({}_R R) = J(R) \subseteq \delta({}_R R)$ , we have the following result.

**Corollary 3.8.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $Z({}_R R)$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a locally projective  $Z({}_R R)$ -cover.
- (3) Every finitely presented  $R$ -module has a locally projective  $Z({}_R R)$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $Z({}_R R)$ -cover.

Since if  $I \leq \text{Soc}({}_R R)$ , then  $R$  is a left PSD ring for  $I$ , and hence we have

**Corollary 3.9.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $\text{Soc}({}_R R)$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a locally projective  $\text{Soc}({}_R R)$ -cover.
- (3) Every finitely presented  $R$ -module has a locally projective  $\text{Soc}({}_R R)$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $\text{Soc}({}_R R)$ -cover.

Next we shall consider the characterizations of  $I$ -semiperfect rings.

**Theorem 3.10.** *Let  $I$  be an ideal of  $R$ . Consider the following conditions:*

- (1) Every finitely generated  $R$ -module has a locally projective  $I$ -cover.
- (2) Every factor module of  ${}_R R$  has a locally projective  $I$ -cover.
- (3) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $I$ -cover.
- (4)  $R$  is  $I$ -semiperfect.
- (5) Every simple  $R$ -module has a locally projective  $I$ -cover.

(6) *Every simple factor module of  ${}_R R$  has a locally projective  $I$ -cover.*

Then  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$  and  $(1) \Rightarrow (5) \Rightarrow (6)$ ,  $(4) \Rightarrow (1)$  if  $R$  is a left PSD ring for  $I$ ; and  $(6) \Rightarrow (4)$  if  $I$  is PSD in  ${}_R R$ .

*Proof.* It follows by Theorem 2.13 and Lemma 3.4.  $\square$

When  $I = \delta({}_R R)$  or  $J(R)$ , we have the following.

**Corollary 3.11.** *The following statements are equivalent for a ring  $R$ .*

- (1)  *$R$  is semiperfect.*
- (2) *Every finitely generated  $R$ -module has a locally projective cover.*
- (3) *Every factor module of  ${}_R R$  has a locally projective cover.*
- (4) *For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective cover.*
- (5) *Every simple  $R$ -module has a locally projective cover.*
- (6) *Every simple factor module of  ${}_R R$  has a locally projective cover.*

**Corollary 3.12.** *The following statements are equivalent for a ring  $R$ .*

- (1)  *$R$  is  $\delta$ -semiperfect.*
- (2) *Every finitely generated  $R$ -module has a locally projective  $\delta$ -cover.*
- (3) *Every factor module of  ${}_R R$  has a locally projective  $\delta$ -cover.*
- (4) *For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $\delta$ -cover.*
- (5) *Every simple  $R$ -module has a locally projective  $\delta$ -cover.*
- (6) *Every simple factor module of  ${}_R R$  has a locally projective  $\delta$ -cover.*

*Proof.* By Lemma 3.4 and Theorem 3.10.  $\square$

**Corollary 3.13.** *The following statements are equivalent for a ring  $R$ .*

- (1)  *$R$  is  $Z({}_R R)$ -semiperfect.*
- (2) *Every finitely generated  $R$ -module has a locally projective  $Z({}_R R)$ -cover.*
- (3) *Every factor module of  ${}_R R$  has a locally projective  $Z({}_R R)$ -cover.*

- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $Z({}_R R)$ -cover.
- (5) Every simple  $R$ -module has a locally projective  $Z({}_R R)$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a locally projective  $Z({}_R R)$ -cover.

**Corollary 3.14.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $\text{Soc}({}_R R)$ -semiperfect.
- (2) Every finitely generated  $R$ -module has a locally projective  $\text{Soc}({}_R R)$ -cover.
- (3) Every factor module of  ${}_R R$  has a locally projective  $\text{Soc}({}_R R)$ -cover.
- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a locally projective  $\text{Soc}({}_R R)$ -cover.
- (5) Every simple  $R$ -module has a locally projective  $\text{Soc}({}_R R)$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a locally projective  $\text{Soc}({}_R R)$ -cover.

#### 4. $I$ -semiregular( $I$ -semiperfect) rings characterized by projectivity classes

Wang [14] gave characterizations of semiregular rings and semiperfect rings by introducing the concept of projectivity classes of modules. Motivated by this, we shall characterize  $I$ -semiregular rings and  $I$ -semiperfect rings in term of projectivity classes of modules in this section.

**Definition 4.1.** ([14]) A class  $\mathcal{P}$  of  $R$ -modules is called a projectivity class if it contains all self-projective modules and for every module  $M$  and every projective module  $P$  with an epimorphism  $f : P \rightarrow M$ ,  $P \oplus M \in \mathcal{P}$  implies that  $M$  is projective.

**Example 4.2.** 1. The class of all quasi-projective modules is a projectivity class.

2. The class of all weakly quasi-projective modules in the sense of Rangaswamy and Vanaja is a projectivity class.

3. The class of all pseudo-projective modules is a projectivity class.

4. The class of all direct-projective modules is a projectivity class.

5. For any perfect ring  $R$ , the class of all discrete  $R$ -modules is a projectivity class.

For a projectivity class  $\mathcal{P}$ , we introduce the following concept.

**Definition 4.3.** Let  $I$  be an ideal of a ring  $R$  and  $M$  be a module. We call an epimorphism  $f : P \rightarrow M$  a  $\mathcal{P}$ -projective  $I$ -cover of  $M$  if  $P \in \mathcal{P}$  and  $\text{Ker} f \leq IP$ ,  $\text{Ker} f$  is PSD in  $P$ .

**Lemma 4.4.** Let  $I$  be an ideal of a ring  $R$ ,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Suppose that  $P$  is a projective module and there is an epimorphism  $f : P \rightarrow M$ . If  $P \oplus M$  has a  $\mathcal{P}$ -projective  $I$ -cover, then  $M$  has a projective  $I$ -cover.

*Proof.* Let  $g : Q \rightarrow P \oplus M$  be a  $\mathcal{P}$ -projective  $I$ -cover of  $P \oplus M$ . We have an exact sequence  $0 \rightarrow g^{-1}(M) \rightarrow Q \xrightarrow{\phi g} P \rightarrow 0$ , where  $\phi : P \oplus M \rightarrow P$  is the projection map. Since  $P$  is projective,  $Q \cong P \oplus g^{-1}(M)$  and  $\text{Ker}(\phi g) = g^{-1}(M)$  is a direct summand of  $Q$ . Note that  $\text{Ker} g$  is PSD in  $Q$ , and so  $\text{Ker} g$  is PSD in  $g^{-1}(M)$  by Lemma 2.2. It is easy to see that  $\text{Ker} g \subseteq Ig^{-1}(M)$ . Clearly, we have an exact sequence  $0 \rightarrow \text{Ker} g \rightarrow g^{-1}(M) \xrightarrow{g} M \rightarrow 0$ . So it suffices to show that  $g^{-1}(M)$  is projective. Since  $P$  is projective with an epimorphism  $f : P \rightarrow M$ , and  $g : g^{-1}(M) \rightarrow M$  is an epimorphism, there is a homomorphism  $\alpha : P \rightarrow g^{-1}(M)$  such that  $g\alpha = f$ , and hence  $\text{Im}\alpha + \text{Ker} g = g^{-1}(M)$ . Since  $\text{Ker} g$  is PSD in  $g^{-1}(M)$ , there is a projective submodule  $L$  of  $\text{Ker} g$  such that  $\text{Im}\alpha \oplus L = g^{-1}(M)$ . Thus  $Q \cong P \oplus \text{Im}\alpha \oplus L$  belongs to  $\mathcal{P}$ . Since  $\mathcal{P}$  is closed under taking direct summands,  $P \oplus \text{Im}\alpha \in \mathcal{P}$ , and there is an epimorphism  $P \rightarrow \text{Im}\alpha$ ,  $\text{Im}\alpha$  is projective. Thus  $g^{-1}(M) = \text{Im}\alpha \oplus L$  is projective, as desired.  $\square$

**Corollary 4.5.** Let  $I$  be an ideal of a ring  $R$ ,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Consider the following statements:

- (1) Every finitely presented  $R$ -module has a  $\mathcal{P}$ -projective  $I$ -cover.
- (2) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $I$ -cover.

- (3) Every cyclically presented left  $R$ -module has a  $\mathcal{P}$ -projective  $I$ -cover.
- (4)  $R$  is  $I$ -semiregular.

Then  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ ,  $(4) \Rightarrow (1)$  if  $R$  is a left PSD ring for  $I$ .

*Proof.* “ $(1) \Rightarrow (2) \Rightarrow (3)$ ” are clear.

“ $(3) \Rightarrow (4)$ ”  $R \oplus R/Rr$  has a  $\mathcal{P}$ -projective by assumption, and hence  $R/Rr$  has a projective  $I$ -cover by Lemma 4.4. Thus  $R$  is  $I$ -semiregular by Theorem 2.10.

“ $(4) \Rightarrow (1)$ ” is clear by Theorem 2.10.  $\square$

**Corollary 4.6.** *Let  $R$  be a ring,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Then the following statements are equivalent.*

- (1)  $R$  is  $Z({}_R R)$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.
- (3) Every finitely presented  $R$ -module has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover..

**Corollary 4.7.** *Let  $R$  be a ring,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Then the following statements are equivalent.*

- (1)  $R$  is  $\text{Soc}({}_R R)$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.
- (3) Every finitely presented  $R$ -module has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.

**Corollary 4.8.** *Let  $\mathcal{P}$  be a projectivity class and which is closed under taking direct summands. The following statements are equivalent for a ring  $R$ :*

- (1)  $R$  is  $\delta$ -semiregular.
- (2) Every cyclically presented left  $R$ -module has a  $\mathcal{P}$ -projective  $\delta$ -cover.
- (3) Every finitely presented  $R$ -module has a  $\mathcal{P}$ -projective  $\delta$ -cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $\delta$ -cover.

When  $\mathcal{P}$  is the class of all direct-projective modules, Corollary 4.8 gives [15, Proposition 4.4].

**Corollary 4.9.** *Let  $\mathcal{P}$  be a projectivity class and which is closed under taking direct summands. The following statements are equivalent for a ring  $R$ :*

- (1)  $R$  is semiregular.
- (2) Every cyclically presented left  $R$ -module has a  $\mathcal{P}$ -projective cover.
- (3) Every finitely presented  $R$ -module has a  $\mathcal{P}$ -projective cover.
- (4) For every finitely generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective cover.

When  $\mathcal{P}$  is the class of all direct-projective modules, Corollary 4.9 gives [17, Corollary 3.4].

**Theorem 4.10.** *Let  $I$  be an ideal of a ring  $R$ ,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Consider the following statements:*

- (1) Every finitely generated  $R$ -module has a  $\mathcal{P}$ -projective  $I$ -cover.
- (2) Every factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $I$ -cover.
- (3) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $I$ -cover.
- (4)  $R$  is  $I$ -semiperfect.
- (5) Every simple  $R$ -module has a  $\mathcal{P}$ -projective  $I$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $I$ -cover.

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) and (1)  $\Rightarrow$  (5)  $\Rightarrow$  (6), (4)  $\Rightarrow$  (1) if  $R$  is a left PSD ring for  $I$ ; and (6)  $\Rightarrow$  (4) if  $I$  is PSD in  ${}_R R$ .

*Proof.* Following by Theorem 2.13 and Lemma 3.4. □

**Corollary 4.11.** *Let  $I$  be an ideal of a ring  $R$ ,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Then the following statements are equivalent.*

- (1)  $R$  is  $Z({}_R R)$ -semiperfect.
- (2) Every finitely generated  $R$ -module has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.

- (3) Every factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.
- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.
- (5) Every simple  $R$ -module has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $Z({}_R R)$ -cover.

**Corollary 4.12.** *Let  $I$  be an ideal of a ring  $R$ ,  $\mathcal{P}$  a projectivity class and which is closed under taking direct summands. Then the following statements are equivalent.*

- (1)  $R$  is  $\text{Soc}({}_R R)$ -semiperfect.
- (2) Every finitely generated  $R$ -module has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.
- (3) Every factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.
- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.
- (5) Every simple  $R$ -module has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $\text{Soc}({}_R R)$ -cover.

**Corollary 4.13.** *Let  $\mathcal{P}$  be any projectivity class and which is closed under taking direct summands. The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is a left  $\delta$ -semiperfect ring.
- (2) Every finitely generated  $R$ -module has a  $\mathcal{P}$ -projective  $\delta$ -cover.
- (3) Every factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $\delta$ -cover.
- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective  $\delta$ -cover.
- (5) Every simple  $R$ -module has a  $\mathcal{P}$ -projective  $\delta$ -cover.
- (6) Every simple factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective  $\delta$ -cover.

**Corollary 4.14.** *Let  $\mathcal{P}$  be any projectivity class and which is closed under taking direct summands. The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is a left semiperfect ring.
- (2) Every finitely generated  $R$ -module has a  $\mathcal{P}$ -projective cover.
- (3) Every factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective cover.



- (4) For every countably generated left ideal  $K$  of  $R$ ,  $R/K$  has a  $\mathcal{P}$ -projective cover.
- (5) Every simple  $R$ -module has a  $\mathcal{P}$ -projective cover.
- (6) Every simple factor module of  ${}_R R$  has a  $\mathcal{P}$ -projective cover.

### 5. $I$ -semiregular( $I$ -semiperfect) rings and $I$ -supplemented modules

It is well known that a ring  $R$  is semiperfect if and only if  $R_R$  is a supplemented module if and only if  ${}_R R$  is a supplemented module. We also know that a ring  $R$  is semiregular if and only if  $R_R$  is a finitely supplemented module if and only if  ${}_R R$  is a finitely supplemented module. Here we introduce the notion of  $I$ -supplemented modules and use it to characterize  $I$ -semiregular(semiperfect) rings.

Let  $R$  be a ring,  $I$  an ideal of  $R$ ,  $M$  a module and  $N, L \leq M$ .  $N$  is called a *supplement* of  $L$  in  $M$  if  $N + L = M$  and  $N$  is minimal with respect to this property. Equivalently,  $M = N + L$  and  $N \cap L \ll N$ . A module  $M$  is called *supplemented* if every submodule of  $M$  has a supplement in  $M$ .  $N$  is called a  $\delta$ -*supplement* [6] of  $L$  if  $M = N + L$  and  $N \cap L \ll_\delta N$ .  $M$  is called a  $\delta$ -*supplemented module* if every submodule of  $M$  has a  $\delta$ -supplement. A module  $M$  is said to be  $\delta$ -*lifting* [6] if for any submodule  $N$  of  $M$ , there exists a direct summand  $K$  of  $M$  such that  $K \leq N$  and  $N/K \ll_\delta M/K$ , equivalently, for every submodule  $N$  of  $M$ ,  $M$  has a decomposition with  $M = M_1 \oplus M_2$ ,  $M_1 \leq N$  and  $M_2 \cap N$  is  $\delta$ -small in  $M_2$ .  $N$  is DM in  $M$  [1] if there is a summand  $S$  of  $M$  such that  $S \leq N$  and  $M = S + X$ , whenever  $N + X = M$  for a submodule  $X$  of  $M$ .  $M$  is DM for  $I$  if any submodule of  $IM$  is DM in  $M$ .  $R$  is a left DM ring for  $I$  if for any finitely generated free left  $R$ -module is DM for  $I$ .

**Definition 5.1.** Let  $R$  be a ring and  $I$  an ideal of  $R$ ,  $M$  a module.  $M$  is called an  $I$ -supplemented module (finitely  $I$ -supplemented module) if for every submodule (finitely generated submodule)  $X$  of  $M$ , there is a projective submodule  $Y$  of  $M$  such that  $X + Y = M$ ,  $X \cap Y \subseteq IY$  and  $X \cap Y$  is DM in  $Y$ .

**Theorem 5.2.** *Let  $R$  be a ring. The following statements are equivalent for a projective module  $M$ .*

- (1)  *$M$  is a  $J(R)$ -supplemented module (a  $\delta(RR)$ -supplemented module, respectively).*
- (2)  *$M$  is a supplemented module (a  $\delta$ -supplemented module, respectively).*

*Proof.* “(1)  $\Rightarrow$  (2)” Let  $M$  be a  $J(R)$ -supplemented module (a  $\delta(RR)$ -supplemented module, respectively). Then for every submodule  $X$  of  $M$ , there is a projective submodule  $Y$  of  $M$  such that  $X + Y = M$ ,  $X \cap Y \subseteq J(R)Y$  ( $X \cap Y \subseteq \delta(RR)Y$ ) and  $X \cap Y$  is DM in  $Y$ . Next we shall show that  $X \cap Y \ll Y$  ( $X \cap Y \ll_{\delta} Y$ ). Assume that  $X \cap Y + L = Y$ ,  $L \leq Y$ . Note that  $X \cap Y$  is DM in  $Y$ , there is a summand  $K$  of  $Y$  which is contained in  $X \cap Y$  such that  $K + L = Y$ . Write  $Y = K \oplus K'$ ,  $K' \leq M$ , then  $\text{Rad}(K) \oplus \text{Rad}(K') = \text{Rad}(Y)$  ( $\delta(K) \oplus \delta(K') = \delta(Y)$ ). Since  $K \subseteq \text{Rad}(Y)$  ( $K \subseteq \delta(Y)$ ),  $K = \text{Rad}(K)$  ( $K = \delta(K)$ ). Since  $K$  is projective,  $K = 0$  ( $K$  is semisimple, thus  $X \cap Y \ll_{\delta} Y$  by [20, Lemma 1.2]), and hence  $L = Y$ , so  $X \cap Y \ll Y$ .

“(2)  $\Rightarrow$  (1)” Let  $M$  be a supplemented module. Then for every submodule  $X$  of  $M$ , there is a submodule  $Y$  of  $M$  such that  $X + Y = M$  and  $X \cap Y \ll Y$ . Since  $M$  is projective,  $Y$  is a direct summand of  $M$ , and hence  $Y$  is projective. It is clear that  $X \cap Y \subseteq \text{Rad}(Y) = J(R)Y$  and  $X \cap Y$  is DM in  $Y$ . (Let  $M$  be a  $\delta$ -supplemented module. Since  $M$  is projective,  $M$  is  $\delta$ -lifting. Thus for every submodule  $X$  of  $M$ , there is a direct summand  $Y$  of  $M$  such that  $M = X + Y$  and  $X \cap Y \ll_{\delta} Y$ . The rest is obvious.)

□

**Theorem 5.3.** *Let  $R$  be a ring and  $I$  an ideal of  $R$ ,  $M$  a projective module. Consider the following conditions:*

- (1)  *$M$  is an  $I$ -supplemented module.*
- (2)  *$M$  is an  $I$ -semiperfect module.*

*Then (1)  $\Rightarrow$  (2), and (2)  $\Rightarrow$  (1) if  $M$  is DM for  $I$ .*

*Proof.* “(1)  $\Rightarrow$  (2)” Let  $M$  be an  $I$ -supplemented module, then for every submodule  $X$  of  $M$ , there is a projective submodule  $Y$  of  $M$  such that  $M = X + Y$ ,  $X \cap Y \subseteq IY$  and  $X \cap Y$  is DM in  $Y$ . We define  $f : Y \rightarrow M/X$  be such that  $f(y) = y + X$ . Then  $f$  is an epimorphism with  $\text{Ker } f = X \cap Y$ , and hence  $Y$  is a projective  $I$ -cover (in the sense of [1]) of  $M/X$ . The rest is obvious by [1, Lemma 3.10].

“(2)  $\Rightarrow$  (1)” Let  $M$  be an  $I$ -semiperfect module, then for every submodule  $X$  of  $M$ , there is a decomposition  $M = A \oplus Y$  such that  $A$  is projective,  $A \subseteq X$  and  $X \cap Y \subseteq IM$ . Thus  $M = X + Y$ ,  $Y$  is projective,  $X \cap Y \subseteq IY$ . Since  $M$  is DM for  $I$ ,  $X \cap Y$  is DM in  $Y$  by [1, Lemma 3.2], as desired.

□

**Corollary 5.4.** *Let  $M$  be a projective module with  $\text{Rad}(M) \ll M$  ( $\delta(M) \ll_\delta M$ ). Then  $M$  is a  $(\delta)$ -supplemented module if and only if  $M$  is a  $(\delta)$ -semiperfect module.*

**Corollary 5.5.** *Let  $R$  be a left DM ring and  $I$  an ideal of  $R$ . Then  $R$  is an  $I$ -semiperfect ring if and only if  ${}_R R$  is an  $I$ -supplemented module.*

Write  $I = J(R)$  or  $\delta({}_R R)$  in Corollary 5.5, since  $R$  is a left DM ring, we have the following.

**Corollary 5.6.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $(\delta)$ -semiperfect;
- (2)  ${}_R R$  is a  $(\delta)$ -supplemented module;
- (3)  $R_R$  is a  $(\delta)$ -supplemented module;
- (4)  ${}_R R$  is a  $J(R)(\delta({}_R R))$ -supplemented module;
- (5)  $R_R$  is a  $J(R)(\delta(R_R))$ -supplemented module.

*Proof.* It follows by Theorem 5.2 and 5.3.

□

Similarly, we obtain the following results.

**Theorem 5.7.** *Let  $R$  be a left DM ring and  $I$  an ideal of  $R$ . Then  $R$  is an  $I$ -semiregular ring if and only if  ${}_R R$  is a finitely  $I$ -supplemented module if and only if  $R_R$  is a finitely  $I$ -supplemented module.*

**Corollary 5.8.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is  $(\delta-)$  semiregular.
- (2)  ${}_R R$  is a finite  $(\delta-)$  supplemented module.
- (3)  $R_R$  is a finite  $(\delta-)$  supplemented module.
- (4)  ${}_R R$  is a finite  $J(R)(\delta({}_R R))$ -supplemented module.
- (5)  $R_R$  is a finite  $J(R)(\delta(R_R))$ -supplemented module.

Since if  $I \subseteq \text{Soc}({}_R R)$ , then  $R$  is a left DM ring, we have

**Corollary 5.9.** *A ring  $R$  is  $\text{Soc}({}_R R)$ -semiperfect if and only if  ${}_R R$  is a  $\text{Soc}({}_R R)$ -supplemented module.*

**Corollary 5.10.** *A ring  $R$  is  $\text{Soc}({}_R R)$ -semiregular if and only if  ${}_R R$  is a finitely  $\text{Soc}({}_R R)$ -supplemented module if and only if  $R_R$  is a finitely  $\text{Soc}(R_R)$ -supplemented module.*

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